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Controllability of Nonlinear Systems of Neutral Type

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1. INTRODUCTION

We consider the controllability of nonlinear perturbations on a bounded interval $J = [1, t_1]$ of the autonomous linear delay system of neutral type

$$\dot{x}(t) = L(x, u), \tag{1}$$

where the operator L is defined by $L(x, u) = A\dot{x}(t-1) + Bx(t-1) + Cx(t) + Du(t) + Hu(t-h)$. We will show that if system (1) is completely controllable then the perturbed system

$$\dot{x}(t) = L(x, u) + f(t, x(\cdot), \dot{x}(\cdot), u(\cdot)) \tag{2}$$

is completely controllable provided the function f, whose domain contains appropriate function spaces, satisfies certain growth and continuity conditions.

For nonautonomous systems without delays this problem has been studied by several authors. For references see Dauer [3]. The approach we will use is to define the appropriate control and its corresponding solution by an integral equation. We then obtain the solution by applying the Schauder fixed point theorem. This approach is used by Dauer and Gahl [3, 4]. In this paper the presence of $\dot{x}(\cdot)$ in the systems (1) and (2) necessitates a function space domain for f as well as an additional continuity assumption in obtaining our main result. We comment also that in system (1) we restrict ourselves to the autonomous case although system (2) may be nonautonomous.

For all vectors and matrices with real entries we will let $|\cdot|$ denote the norm obtained by adding the absolute values of each component or each entry. We will also use Lebesgue measure and integration.

2. PRELIMINARIES

The vector function x has its values in R^n the space of real n-tuples, and the control function u has its values in R^m. The constant matrices A, B, C, D, and H, have the appropriate dimensions, and h is a positive constant. We take the

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class of *admissible control functions* for systems (1) and (2) to be all functions u with $u(t) = 0$ for $t < 1$, which are also piecewise continuous and bounded on J with all points of discontinuity contained in the set

$$S_1 = \{1, t_1 - 1, t_1 - 2, \ldots, t_1 - h, t_1 - h - 1, t_1 - h - 2, \ldots\} \cap J$$

and being of simple type yielding finite jumps in u. The structure of S_1 is necessary because of the special way in which the appropriate control function must be defined in order to establish the controllability of system (2) on J.

Let ϕ be the initial function for systems (1) and (2) and be of class C^1 on $[0, 1]$. We will use the results established by Bellman and Cooke [2, Sect. 6.6] in order to obtain the unique solution $x_L(t; u)$ of system (1) on J satisfying $x_L(t; u) = \phi(t)$ for $t \in [0, 1]$ and corresponding to u, an admissible control function. Let $X(t)$ be the unique $n \times n$ matrix function with the following properties:

(a) $X(t) = 0$ for $t < 0$,

(b) $X(0) = I$, the identity matrix

(c) $X(t) - AX(t - 1)$ is continuous on $[0, \infty)$,

(d) $X(t)$ satisfies $\dot{X}(t) = L(X, 0)$ for $t \in (0, \infty) - S_2$, where S_2 is the set of nonnegative integers.

Note that $X(t)$ may have jump discontinuities at points in S_2 but is of class C^1 on $(0, \infty) - S_2$, and $\dot{X}(t)$ has finite left and right limits on S_2. Then [2, Theorem 6.4] we have

$$\begin{aligned} x_L(t; u) = {} & X(t - 1)\,\phi(1) - X(t - 2)\,A\phi(1) \\ & + \int_0^1 X(t - s - 1)\,[A\dot{\phi}(s) + B\phi(s)]\,ds \\ & + \int_1^t X(t - s)\,[Du(s) + Hu(s - h)]\,ds, \end{aligned} \tag{3}$$

for $t \in J$. This expression was obtained in [2] formally by using the Laplace transform, and by direct computation it may be verified as a continuous function which satisfies (1) on J except for a finite number of points. It will follow from the proof of Theorem 1 in Section 3 that these points are contained in the set

$$S_3 = S_2 \cup \{t\colon t = t_1 \pm h \pm k,\ t_1 \pm k, \text{ or } h \pm k, \text{ for } k \in S_2\}.$$

Now define a matrix function Z by

$$Z(t, s) = X(t - s)\,D + X(t - s - h)\,H.$$

Using the fact that $u(t) = 0$ for $t < 1$ and $X(t) = 0$ for $t < 0$, it follows from Eq. (3) that

$$x_L(t; u) = x_L(t; 0) + \int_1^t Z(t, s)\,u(s)\,ds, \qquad \text{for } t \in J.$$

For future reference we note that for fixed t, $Z(t, s)$ is piecewise continuous in s with finite jumps at $s = t, t - 1, t - 2, \ldots, t - h, t - h - 1, t - h - 2, \ldots$, and hence $Z(t_1, s)$ is piecewise continuous in s on J with finite jumps at points in S_1. Also since $\dot{X}(t)$ is continuous and bounded on $(0, t_1) - S_2$, $\partial Z(t, s)/\partial t = \dot{X}(t - s) D + \dot{X}(t - s - h) H$ is continuous and bounded for $t - s \notin S_2$, $t - s - h \notin S_2$, and $1 \leqslant s \leqslant t \leqslant t_1$. From Eq. (3) it also follows that $\dot{x}_L(t; 0)$ is continuous and bounded on $J - S_2$.

3. Controllability Results

We say system (2) is *completely controllable* on J in case for every function ϕ which is of class C^1 on $[0, 1]$ and every $x_1 \in R^n$ there exists an admissible control function u such that a solution of

$$\begin{aligned} \dot{x}(t) &= L(x, u) + f(t, x(\cdot), \dot{x}(\cdot), u(\cdot)) && \text{for } t \in J, \\ x(t) &= \phi(t) && \text{for } t \in [0, 1], \end{aligned}$$

satisfies $x(t_1) = x_1$.

We take $f: J \times M_1 \times M_2 \times M_3 \to R^n$, where M_1 is the space of continuous n-vector functions on $[0, t_1]$, M_2 is the space of continuous n-vector functions on $[0, t_1] - S_3$ with finite right and left limits on S_3, and M_3 is the space of admissible control functions on $[1 - h, t_1]$. We also assume that for each $(x, y, u) \in M_1 \times M_2 \times M_3$, $f(t, x(\cdot), y(\cdot), u(\cdot))$ is a continuous function of t on $J - S_3$.

It follows from Section 2 that any solution $x(t; u)$ of system (2) corresponding to an admissible control function u satisfies

$$x(t; u) = x_L(t; u) + \int_1^t X(t - s) f(s, x(\cdot; u), \dot{x}(\cdot; u), u(\cdot))\, ds$$

for $t \in J$, so that $x(t; u)$ satisfies (2) on $J - S_3$.

We now define the matrix

$$W = \int_J Z(t_1, s)\, Z^*(t_1, s)\, ds,$$

where Z^* denotes the transpose matrix of Z.

The following proposition on the controllability of system (1) is similar to corresponding results for linear control systems of various types including some with delays and some without as discussed by Dauer and Gahl [4]. The proof follows as the one given for the proposition in [4] and is omitted.

PROPOSITION 1. *The system* (1) *is completely controllable on J if and only if W is nonsingular.*

We are now ready to obtain our main result which extends those of Dauer [3] to perturbations of neutral systems. Let Q be the Banach space of all functions (x, u): $[0, t_1] \times [1 - h, t_1] \to R^n \times R^m$, where x is in M_1, $\dot{x}$ is in M_2, and u is in M_3. The norm on Q is

$$\|(x, u)\| = \| x \| + \| u \| + \| \dot{x} \|,$$

where

$$\| x \| = \sup | x(t)| \qquad \text{for } t \in [0, t_1],$$
$$\| \dot{x} \| = \sup | \dot{x}(t)| \qquad \text{for } t \in [0, t_1] - S_3,$$

and

$$\| u \| = \sup | u(t)| \qquad \text{for } t \in [1 - h, t_1].$$

That Q is a Banach space follows with the aid of the fact that if $\{x_n(t)\}$ is a uniformly convergent sequence on a bounded interval converging to some function $x(t)$ and if $\{\dot{x}_n(t)\}$ is also uniformly convergent, then $\{\dot{x}_n(t)\}$ converges uniformly to $\dot{x}(t)$.

We now assume the following conditions on f:

(I) $\sup\{| f(t, x(\cdot), \dot{x}(\cdot), u(\cdot))| : t \in J - S_3\} \leqslant G(r) < \infty$ for all $(x, u) \in Q$ such that $\|(x, u)\| \leqslant r$, where $\lim_{r\to\infty} G(r)/r = 0$.

(II) Let $\{(x_n, u_n)\}$ be a bounded sequence in Q such that $\{u_n(t - h)\}$, $\{u_n(t)\}$, $\{x_n(t)\}$, and $\{x_n(t - 1)\}$ are each equicontinuous for $t \in [a, b] \subseteq J$. Then $\{f(t, x_n(\cdot), \dot{x}_n(\cdot), u_n(\cdot))\}$ is equicontinuous on $[a, b]$.

A discussion of examples of functions f satisfying the growth condition (I) and the continuity condition (II) follows the proof of Theorem 1.

THEOREM 1. *Let f be continuous on $J - S_3 \times M_1 \times M_2 \times M_3$ with finite left and right limits on S_3, and assume f satisfies conditions* (I) *and* (II). *If system* (1) *is completely controllable on J, then system* (2) *is completely controllable on J.*

Proof. By Proposition 1, W^{-1} exists. Let ϕ be an n-vector function on [0, 1] of class C^1, and let $x_1 \in R^n$. We define an operator T on Q by $T(x, u) = (y, v)$, where

$$v(t) = Z^*(t_1, t)\, W^{-1} \left[x_1 - x_L(t_1; 0) - \int_J X(t_1 - s) f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds\right] \quad \text{for } t \in J,$$

and

$$v(t) = 0 \qquad \text{for } t < 1;$$
$$y(t) = x_L(t; 0) + \int_1^t Z(t, s)\, v(s)\, ds + \int_1^t X(t - s) f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds \quad \text{for } t \in J,$$

and

$$y(t) = \phi(t) \qquad \text{for } 0 \leqslant t < 1.$$

We show that $T: Q \to Q$. By the definition of Z and the remarks following in Section 2, $v(t)$ is an admissible control function. Since $x_L(1; 0) = \phi(1)$, $y(t)$ is continuous on $[0, t_1]$. In order to show that $(y, v) \in Q$ it remains to establish that $\dot{y}(t) \in M_2$. Let $t \in J - S_3$. In obtaining an expression for $\dot{y}(t)$ we use the fact that $Z(t, s)$ is discontinuous on J for $s = t, t - 1, \ldots, t - h, t - h - 1, \ldots$, and $X(t - s)$ is discontinuous for $s = t, t - 1, \ldots$. Then

$$\int_1^t Z(t, s)\, v(s)\, ds = \sum_{j=1}^{p(t)} \left(\int_{t-a_j}^{t-a_{j+1}} Z(t, s)\, v(s)\, ds \right) + \int_1^{t-a_1} Z(t, s)\, v(s)\, ds,$$

where $t - a_j$, $j = 1, \ldots, p(t) + 1$, are the points of discontinuity in s of $Z(t, s)$ for $s \in [1, t]$. Using $(-)$ and $(+)$ to denote left and right limits, respectively,

$$\begin{aligned} \frac{d}{dt} \int_1^t & Z(t, s)\, v(s)\, ds \\ &= \int_1^t \frac{\partial Z}{\partial t}(t, s)\, v(s)\, ds + \sum_{j=1}^{p(t)} [Z(t, (t - a_{j+1})^-) \\ &\quad \cdot v(t - a_{j+1}) - Z(t, (t - a_j)^+) \cdot v(t - a_j)] + Z(t, (t - a_1)^-)\, v(t - a_1). \end{aligned}$$

We are also using here the fact that the points $t - a_j$ are points of continuity of v, since if $t - a_j \in S_1$ then $t \in S_3$. By the definition of $Z(t, s)$

$$\frac{d}{dt} \int_1^t Z(t, s)\, v(s)\, ds = \int_1^t (\dot{X}(t - s)\, D + X(t - s - h)\, H)\, v(s)\, ds + \sum_{j=0}^{p(t)} R_j(t),$$

where

$$\begin{aligned} R_j(t) = {} & [X(a_{j+1}^+)\, D + X((a_{j+1} - h)^+)\, H]\, v(t - a_{j+1}) \\ & - [X(a_j^-)\, D + X((a_j - h)^-)\, H]\, v(t - a_j), \end{aligned}$$

$a_0 = 0$, and recall that $X(0^-) = 0$. We note that a similar derivation is needed to show that the expression for $x_L(t; u)$ in Eq. (3) satisfies (1) on J except for the points where $t - a_j \in S_1$ or $t \in S_2$. Thus the defined structure of S_3 is necessary.

Similarly, again for $t \in J - S_3$,

$$\begin{aligned} \frac{d}{dt} \int_1^t & X(t - s)\, f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds \\ &= \int_1^t \dot{X}(t - s)\, f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds \\ &\quad + \sum_{i=1}^{[t]-1} [X((i - 1)^+)\, f(t - i + 1, x(\cdot), \dot{x}(\cdot), u(\cdot)) \\ &\quad - X(i^-)\, f(t - i, x(\cdot), \dot{x}(\cdot), u(\cdot))] \\ &\quad + X(([t] - 1)^+)\, f(t - [t] + 1, x(\cdot), \dot{x}(\cdot), u(\cdot)), \end{aligned}$$

where $[t]$ is the largest integer $\leqslant t$ and if $[t] = 1$, the sum is empty. We are also using here the fact that if $t \in J - S_3$ then $t - i \in J - S_3$ for $i = 1, 2, \ldots, [t] - 1$, along with the continuity assumption on f. Upon rearranging terms we have

$$\frac{d}{dt} \int_1^t X(t-s) f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds = \int_1^t \dot{X}(t-s) f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds + \sum_{i=0}^{[t]-1} P_i(t),$$

where

$$P_i(t) = [X(i^+) - X(i^-)]\, f(t-i, x(\cdot), \dot{x}(\cdot), u(\cdot)).$$

Then for $t \in J - S_3$ we have

$$\dot{y}(t) = \dot{x}_L(t; 0) + \int_1^t (\dot{X}(t-s)\, D + \dot{X}(t-s-h)\, H)\, v(s)\, ds + \sum_{j=0}^{\nu(t)} R_j(t) + \int_1^t \dot{X}(t-s) f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds + \sum_{i=0}^{[t]-1} P_i(t). \tag{4}$$

Since $\dot{y}(t)$ is continuous on $J - S_3$ and has finite left and right limits on S_3, $\dot{y}(t) \in M_2$ and $T: Q \to Q$.

In what follows let

$$\begin{aligned}
a_1 &= \sup |Z(t, s)| && \text{for } 1 \leqslant s \leqslant t \leqslant t_1, \\
a_2 &= |W^{-1}|, \\
a_3 &= \sup(|x_L(t; 0)| + |x_1|) && \text{for } t \in J, \\
a_4 &= \sup |X(t)| && \text{for } 0 \leqslant t \leqslant t_1, \\
a_5 &= \sup |\dot{X}(t)| && \text{for } t \in [0, t_1] - S_2, \\
a_6 &= \sup |\dot{x}_L(t; 0)| && \text{for } t \in J - S_2, \\
a_7 &= |D| + |H|, \\
b &= \max\{(t_1 - 1)\, a_1, 1, (t_1 - 1)\, a_5 a_7 + 2(p(t_1) + 1)\, a_4 a_7\}, \\
c_1 &= 6 b a_1 a_2 a_4 (t_1 - 1), \\
c_2 &= 6 a_4 (t_1 - 1), \\
c_3 &= 6[2[t_1]\, a_4 + (t_1 - 1)\, a_5], \\
d_1 &= 6 a_1 a_2 a_3 b, \\
d_2 &= 6 a_3, \\
d_3 &= 6 a_6, \\
c &= \max\{c_1, c_2, c_3\}, \\
d &= \max\{d_1, d_2, d_3\}, \\
\sup |f| &= \sup |f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))| && \text{for } s \in J - S_3.
\end{aligned}$$

Since ϕ is of class C^1 on $[0, 1]$ and $\lim_{r\to\infty} G(r)/r = 0$, there exists $r > 0$ such that $\sup |\phi(t)|$ for $t \in [0, 1] \leqslant r/3$, $\sup |\dot{\phi}(t)|$ for $t \in [0, 1] \leqslant r/3$, $G(r)/r < 1/2c$, and $r \geqslant 2d$. Then $cG(r) + d \leqslant (r/2) + (r/2) = r$. From property (I) it follows that if $\|(x, u)\| \leqslant r$, then $c \sup |f| + d \leqslant r$. Now let

$$Q(r) = \{(x, u) \in Q \colon \|(x, u)\| \leqslant r\},$$

and let $(x, u) \in Q(r)$. Then

$$\begin{aligned} | v(t)| &\leqslant a_1 a_2[a_3 + (t_1 - 1)\, a_4 \sup |f|] \leqslant (d_1/6b) + (c_1/6b) \sup |f| \\ &\leqslant (1/6b)\, [d + c \sup |f|] \leqslant (r/6b) \leqslant (r/6) \qquad \text{for all } t \in J. \end{aligned}$$

Similarly

$$\begin{aligned} | y(t)| &\leqslant a_3 + (t_1 - 1)\, a_1 \| v \| + (t_1 - 1)\, a_4 \sup |f| \\ &\leqslant (d_2/3) + b \| v \| + (c_2/6) \sup |f| \\ &\leqslant (r/6) + \tfrac{1}{6}[d + c \sup |f|] \leqslant (r/3) \qquad \text{for all } t \in J. \end{aligned}$$

Using Eq. (4) and the expressions for $R_j(t)$ and $P_i(t)$ and letting $t \in J - S_3$,

$$\begin{aligned} | \dot{y}(t)| &\leqslant a_6 + (t_1 - 1)\, a_5 a_7 \| v \| + 2(p(t_1) + 1)\, a_4 a_7 \| v \| \\ &\quad + ((t_1 - 1)\, a_5 + 2[t_1]\, a_4) \sup |f| \\ &\leqslant (d_3/6) + b \| v \| + (c_3/6) \sup |f| \\ &\leqslant (r/6) + \tfrac{1}{6}[d + c \sup |f|] \leqslant r/3. \end{aligned}$$

Hence $\|(y, v)\| \leqslant r$, and $T \colon Q(r) \to Q(r)$.

By the continuity assumption on f, T is continuous on Q. Now let $P(r)$ be the closed convex hull of $T(Q(r))$, and we show that T has a fixed point in $P(r)$. Since $Q(r)$ is closed, bounded, and convex in Q and $T \colon Q(r) \to Q(r)$; $P(r)$ is also closed, bounded, and convex in $Q(r)$ and $T \colon P(r) \to P(r)$. In order to apply the Schauder fixed point theorem it remains to show that T is completely continuous on $P(r)$.

Let $\{(y_n, v_n)\}_{n=1}^{\infty}$ be a sequence in $T(P(r))$, and we show it has a convergent subsequence in Q. Let $(y_n, v_n) = T(x_n, u_n)$ for some $(x_n, u_n) \in P(r)$. Since $\|(y_n, v_n)\| \leqslant r$ for all n, the sequences $\{y_n(t)\}$, $\{\dot{y}_n(t)\}$, and $\{v_n(t)\}$ are each uniformly bounded on the sets $[0, t_1]$, $[0, t_1] - S_3$, and $[1 - h, t_1]$ respectively. Further, by the boundedness assumption on f and since $P(r)$ is contained in $Q(r)$; the sequence $\{y_n(t)\}$ is equicontinuous on $[0, t_1]$. Also since $Z^*(t_1, t)$ has a continuous extension on the closure of each subinterval of $[1 - h, t_1] - S_1$, the sequence $\{v_n(t)\}$ can be extended to be equicontinuous on the closure of each of these subintervals. We note that the equicontinuity holds for all members of $T(Q(r))$. Hence it holds for the limit points in Q of all convex combinations of members of $T(Q(r))$ and, therefore, it holds for all members of $P(r)$. Thus $\{x_n(t)\}$ is equicontinuous on $[0, t_1]$, and $\{u_n(t)\}$ can be extended to be equicontinuous on the closure of each subinterval of $[1 - h, t_1] - S_1$.

We now apply property (II) in order to show that $\{\dot{y}_n(t)\}$ can be extended to be equicontinuous on the closure of each subinterval of $[0, t_1] - S_3$. For $t \in J - S_3$, it follows that $t - i \in [0, t_1] - S_3$ and $t - h - i \in [1 - h, t_1] - S_1$, for $i = 0, 1,..., [t]$. Therefore, each of the sequences $\{u_n(t - i)\}$, $\{u_n(t - h - i)\}$, $\{x_n(t - i)\}$, and $\{x_n(t - i - 1)\}$ for $i = 0, 1,..., [t] - 1$, can be extended to be equicontinuous on the closure of each subinterval of $J - S_3$. By property (II) the sequence $\{f(t - i, x_n(\cdot), \dot{x}_n(\cdot), u_n(\cdot))\}$ is equicontinuous on the closure of each of these subintervals for $i = 0, 1,..., [t] - 1$. It follows from Eq. (4) that the sequence $\{\dot{y}_n(t)\}$ can be extended to be equicontinuous on the closure of each subinterval of $[0, t_1] - S_3$. Now by successive applications of Ascoli's theorem we obtain a convergent subsequence in Q of $\{(y_n, v_n)\}$. Hence $T(P(r))$ is sequentially compact and, therefore, its closure is also sequentially compact. This implies that T is completely continuous on $P(r)$.

By the Schauder fixed-point theorem T has a fixed point in $P(r)$, call it (x, u). Therefore

$$x(t) = x_L(t; 0) + \int_1^t Z(t, s)\, u(s)\, ds + \int_1^t X(t - s)\, f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds$$

$$\text{for } t \in J,$$

and

$$x(t) = \phi(t) \qquad \text{for } t \in [0, 1].$$

Also

$$\begin{aligned} x(t_1) = x_L(t_1; 0) &+ \int_J Z(t_1, s)\, Z^*(t_1, s)\, W^{-1}(x_1 - x_L(t_1; 0) \\ &- \int_J X(t_1 - s)\, f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds)\, ds \\ &+ \int_J X(t_1 - s)\, f(s, x(\cdot), \dot{x}(\cdot), u(\cdot))\, ds = x_1 . \end{aligned}$$

This completes the proof, since $x(t)$ is the desired solution.

We remark that condition (II) was necessary in order to establish the complete continuity of the operator T. We now discuss some examples of functions f which satisfy the hypotheses of Theorem 1.

EXAMPLE 1. $f = k(t, x(t), x(t - 1), u(t), u(t - h)) + \int_1^t g(s, x(s), x(s - w(s)), \dot{x}(s), \dot{x}(s - \theta(s)), u(s), u(s - \lambda(s))\, ds$, where, w, θ, and λ are nonnegative measurable functions; and the n-vector functions k and g are defined on the appropriate R^n spaces, are continuous, and satisfy a growth condition on R^n spaces as in (I). We remark that terms of the form $\int_1^t g(x(s))\, ds$ are of so called *renewal* type and are included in systems treated by Alekal *et al.* [1] and Dauer and Gahl [4].

EXAMPLE 2. Take $g = 0$ in Example 1. Then note that f is independent of $\dot{x}(\cdot)$ and has an R^n space domain, which is characteristic of perturbation functions used in other treatments (see Dauer and Gahl [3, 4]).

We remark that if f depends on $\dot{x}(\cdot)$ then in order to satisfy condition (II), we must include function spaces in the domain of f. To see this suppose $f\colon J \times R^n \times R^n \to R^n$ with $\{f(t, x_n(t), \dot{x}_n(t))\}$ equicontinuous whenever $\{x_n(t)\}$ and $\{\dot{x}_n(t)\}$ are both uniformly bounded on J and $\{x_n(t)\}$ is equicontinuous. Then, for arbitrary t_0, y_1, y_2, and y_3, the sequences $x_n(t) = (-2/n\pi)(y_1 - y_2) \times \cos[(n\pi/2)(t - t_0)] + y_2(t - t_0) + y_3$, $t_n = t_0 + 1/n$, and $s_n = t_0 + 2/n$, defined for $n = 1, 2, \ldots$, have the property that $\{f(t_n, x_n(t_n), \dot{x}_n(t_n))\}$ converges to $f(t_0, y_3, y_1)$, and $\{f(s_n, x_n(s_n), \dot{x}_n(s_n))\}$ converges to $f(t_0, y_3, y_2)$. By the equicontinuity assumption it follows that $f(t_0, y_3, y_1) = f(t_0, y_3, y_2)$, which implies that $f(t, x, \dot{x})$ is independent of the $\dot{x}$ argument.

EXAMPLE 3. (a) $f = k(t, x(t), x(t-1), y(t), u(t-h)) \cdot g(\| \dot{x} \|)$, where k is as in Example 1; and g is real valued, continuous, and bounded.

(b) Take f as in (a) except that k is continuous and bounded; and g is continuous with $\lim_{r\to\infty}(g(r)/r) = 0$.

(c) $f = k(t, x(t), x(t-1), u(t), u(t-h)) + g(\| \dot{x} \|)$, where k is as in Example 1, and g is as in (b).

EXAMPLE 4. $f = \sum_{i=1}^{k} h_i(t, x(t), x(t-1), \dot{x}(t_i)\, u(t), u(t-h))$, where each h_i is continuous, satisfies a growth condition as in (I), and each $t_i \in J$.

We remark that in Examples 3 and 4 system (2) is no longer strictly a delay system, since the argument in $\dot{x}(\cdot)$ may be greater than t.

We now obtain a sufficient condition of an algebraic nature for system (1) to be completely controllable on J. This condition will be easier to check than the condition of Proposition 1.

THEOREM 2. *If* $\operatorname{rank}[D, CD] = n$, *then system* (1) *is completely controllable on* J.

Proof. We show W is nonsingular. Suppose not. Then there exists an n-vector $v \neq 0$ such that $vWv^* = 0$. Then $\int_J [vZ(t_1, s)][vZ(t_1, s)]^* \, ds = 0$, and hence $vZ(t_1, s) = 0$ on J except possibly for a finite number of points. In particular $vZ(t_1, s) = 0$ for all s in a neighborhood to the left of t_1. Therefore $vZ(t_1, t_1^-) = 0$, and $v(\partial Z(t_1, t_1^-)/\partial s) = 0$. Using the fact that $Z(t, s) = X(t-s)\, D + X(t-s-h)\, H$, $X(0^+) = I$, and $X(t) = 0$ for $t < 0$, it follows that $vD = 0$ and $v[-\dot{X}(0^+)\, D - \dot{X}((-h)^+)\, H] = v[-CD] = 0$. Hence, $\operatorname{rank}[D, CD] < n$, a contradiction.

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